1 Malthus: The curse of fixed factors

Why have we seen close to no economic growth until the mid 17^{th} century? One of the earliest economists thinking about this question, and income growth in general, was the English economist Thomas Malthus. In particular, he asked why income per person was close to stagnant in the medieval UK. In Malthus (1798) he answered this question by a theory of endogenous population growth. His idea was that whenever land became more productive, the population would increase and food production per person would remain constant in the long run. One particular example is the introduction of the potato in Ireland after 1750. A potato field produces two to three times more nutrition than a weed field (Ireland becomes more productive). After some time, the population of Ireland tripled, and living standards remained unaltered. In this chapter, we will formalize this idea and ask ourselves what could be different today such that permanent economic growth has become possible.

1.1 Modeling the economy of medieval UK

In the medieval UK, the vast majority of people worked in food production. Hence, we will abstract from other economic activities, such as craftsmanship, and assume the entire economy produces a single output good Y (food). Though households used some capital in food production, such as livestock and plows, the main factors of production were land and people, and we will restrict our analysis to those. For the sake of food production, labor was relatively homogeneous, and we assume we can aggregate it into a single measure L that may change over time. Moreover, we assume that the amount of land, X, is fixed. Finally, we have to think about the level of technology, B, that people use when producing food on the land. To give an example, the aforementioned crops are a level of technology. Weed produces fewer nutrition per acre of land than potato, i.e., using potato is a better technology.

Given these thoughts, we model the production of the output good as

$$Y(t) = B(t)X^{\alpha}L(t)^{1-\alpha}, \tag{1}$$

where $\alpha < 1$ is the relative importance of land in the production process. To make the notation more compact, we can write this as

$$Y(t) = A(t)L(t)^{1-\alpha}, \tag{2}$$

with $A(t) = B(t)X^{\alpha}$ being the efficient land. Hence, when land becomes three times more productive, $A(t)$ scales up by a factor of three. The marginal product of labor is

$$\frac{\partial Y(t)}{\partial L(t)} = (1-\alpha)AL(t)^{-\alpha} > 0, \tag{3}$$

i.e., adding more labor always increases total output. For example, more people working a weed field allows for a better plowing, protection from crows, and a better defense against robbers. Importantly, these marginal returns become smaller as we increase labor, i.e., the production function features diminishing marginal returns to labor:

$$\frac{\partial^2 Y(t)}{\partial^2 L(t)} = -\alpha(1-\alpha)AL(t)^{-\alpha-1} < 0. \tag{4}$$

Diminishing marginal returns appears a logical implication. Certainly, three people can plow a single field better than two people, yet the additional gain the third person brings is smaller than the additional gains that the second person has brought to the production process. As we keep increasing labor, the additional labor may no longer plow the field on a regular basis but only substitute workers when they become sick, i.e., their marginal return is still positive but small. A direct implication of diminishing marginal returns is that output per worker, $y(t) = \frac{Y(t)}{L(t)} = AL(t)^{-\alpha}$, is decreasing in the amount of labor:

$$\frac{\partial y(t)}{\partial L(t)} = -\alpha AL(t)^{-\alpha-1} < 0. \tag{5}$$

So far, we consider only the production side of the economy. For households' decisions, what matters is their income. Here, we will just assume that each farm is family owned and all output goes to the household, i.e., income equals production. Given that we know household income, we can now turn to a theory of endogenous population growth which lies at the heart of Malthus theory is a theory of endogenous population growth. Malthus postulates that there exists a natural birth rate, Z. For example, if every couple could have on average 8 children, then the natural birth rate would be 300% per generation. According to Malthus, what leads people to have fewer (surviving) children than the natural birth rate is too low income. Low income leads to famines, diseases, and wars thus reducing the population growth rate. Accordingly, we model the population growth rate as increasing in income (output) per person:

$$n(t) = \frac{\dot{L}(t)}{L(t)} = Z - \frac{1}{y(t)}. \tag{6}$$

Note, as $y(t) \to \infty$, population growth approaches Z.

1.2 Constant technology level

We start by assuming a constant level of technology with $A(t) = A$. Our analysis will proceed in two steps. First, we will study the steady state of the economy, that is, a state where an endogenous variable of the model is constant over time. Afterward, we will study the transition dynamics of the model when it is not in steady state, i.e., we will study whether and how the economy converges to its steady state.

1.2.1 The steady state

As we are interested in the question why output per person is constant over time, for the model to be useful, the model should be such that there exists a steady state in output per person. Assume that such a steady state exists, i.e., its growth rate is zero. To obtain the growth rate of output per capita, we start with log output per capita, and take the derivative with respect to time and use the fact that the derivative of a variable in logs with respect to time is the growth rate of that variable:

$$y(t) = AL(t)^{-\alpha} \tag{7}$$

$$\ln y(t) = \ln A - \alpha \ln L(t) \tag{8}$$

$$\frac{\dot{y}(t)}{y(t)} = -\alpha \frac{\dot{L}(t)}{L(t)} = 0 \tag{9}$$

Now substitute in the law of motion for labor:

$$\frac{\dot{y}(t)}{y(t)} = -\alpha \frac{\dot{L}(t)}{L(t)} = 0 \tag{10}$$

$$-\alpha\left(Z - \frac{1}{y^*}\right) = 0 \tag{11}$$

$$y^* = \frac{1}{Z}. \tag{12}$$

The steady state indeed exists as the endogenous variable y depends only on exogenous parameters (constants). Once output per person equals $1/Z$, output per person will be constant at that level going forward.

The steady state has two important implications. First, countries with a higher natural population growth rate have lower output per person in steady state. Second, output per person in steady state does not depend on the productivity of the economy, A. To understand the intuition for the result better, it is instructive to solve for the amount of labor in the steady state using the production function:

$$y^* = \frac{1}{Z} \tag{13}$$

$$A\left(L^*\right)^{-\alpha} = \frac{1}{Z} \tag{14}$$

$$L^* = (AZ)^{\frac{1}{\alpha}} \tag{15}$$

The steady state population increases in the natural birth rate and the amount of efficient land. Countries with more land or a better technology to work that land will have larger populations in the long run. Given this result, it is easy to see why a higher A does not increase output per worker in steady state: Though the economy has a better technology, it has also more people which offsets the better technology. It is worth reflecting on the key model ingredients delivering this result. First, there is the endogenous population growth that rises whenever output per worker is above its steady state level. Second, we have diminishing marginal returns to labor which implies that the additional labor reduces output per worker. However, we will see in a later chapter that endogenous population growth and diminishing marginal returns to labor by themselves are not enough to deliver the result. Instead, it is the combination of those and a second factor of production that is exogenous, effective land. Foreshadowing the later results, one can see the intuition for the importance of the fixed factor already here. When the population grows, land becomes more productive (its marginal product increases) creating incentives to increase its quantity. If land could grow together with labor, output per worker could grow. It is the assumption that land is fixed that prevents this intuition to manifest in the Malthus model.

1.2.2 The dynamics of the model

The steady state is just one possible level of output per worker. Several basic questions remain open: How does an economy behave that is not in steady state? Should we expect an economy to converge to its steady state over time? To answer these questions, we need to study the dynamics of the economy without imposing that it is in steady state. Here, a general solution tells us the level of output per worker, $y(t)$, for an initial starting point, $y(0)$, and the time that has passed, t. To find such a solution, we go back to the dynamics of output per worker over time:

$$\frac{\dot{y}(t)}{y(t)} = -\alpha \frac{\dot{L}(t)}{L(t)} \tag{16}$$

The equation makes intuitive sense: Output per worker growth is negatively proportional to the growth rate in labor. As more workers arrive, the fixed factor land loses productivity. Now substitute the law of motion for labor to obtain a first-order differential equation in $y(t)$:

$$\frac{\dot{y}(t)}{y(t)} = -\alpha \left[Z - \frac{1}{y(t)} \right] \tag{17}$$

$$\dot{y}(t) = -\alpha Z y(t) + \alpha. \tag{18}$$

This equation is similar to the type of equations we have analyzed before ($\dot{Y}(t) = g_Y Y(t)$) but for the constant α. To deal with it, we use a simple trick of defining an auxilary variable, $u(t)$:

$$u(t) = \dot{y}(t) = -\alpha Z y(t) + \alpha \tag{19}$$

$$\dot{u}(t) = -\alpha Z \dot{y}(t) \tag{20}$$

$$\Rightarrow \dot{u}(t) = -\alpha Z u(t). \tag{21}$$

To this type of equations, we have seen that the solution is given by an exponential growth process:

$$u(t) = u(0) \exp(-\alpha Z t). \tag{22}$$

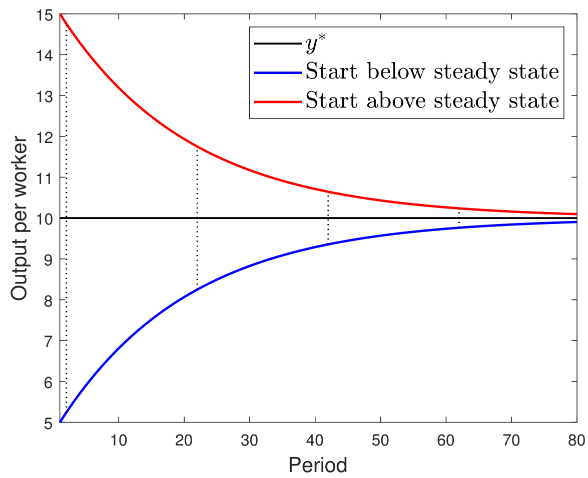
This equation tells us how $u(t)$ behaves over time. Obviously, we are not interested in the behavior of $u(t)$ but of the behavior of $y(t)$. To obtain it, we substitute back the definition of $u(t)$ and solve for $y(t)$:

$$-\alpha Z y(t) + \alpha = [-\alpha Z y(0) + \alpha] \exp(-\alpha Z t) \tag{23}$$

$$y(t) = \underbrace{\frac{1}{Z}}_{y^*} + \left[y(0) - \frac{1}{Z} \right] \exp(-\alpha Z t). \tag{24}$$

This equation is what we are looking for.For any initial $y(0)$ we can determine $y(t)$ for any period t. Three observations are in order. First, when the economy starts in steady state, $y(0) = \frac{1}{Z}$, $y(t) = \frac{1}{Z}$ $\forall t$. This point is obvious: Once the economy is in its steady state, it stays in its steady state. If the solution had said something different, we would have made a mistake. Second, and less obvious, when the economy starts above its steady state, $y(0) > y^*$, we have that it stays above steady state, $y(t) > y^*$, $\forall t$. The reverse is true when it starts below steady state. However, and third, as $t \mapsto \infty$, i.e., as time passes, the economy converges to its steady state for any starting point: $\exp(-\alpha Z t) \mapsto 0$ and, hence, $y(t) \mapsto y^*$. This feature makes the steady state a particularly interesting point to analyze as we can interpret it as the long run outcome of any economy. As discussed above, in the Malthus model, long run income per person always converges to $1/Z$, i.e., all economies converge to the same level of income per person (assuming that the natural birth rates are the same across economies which sounds reasonable).

Figure 1: Output per worker over time



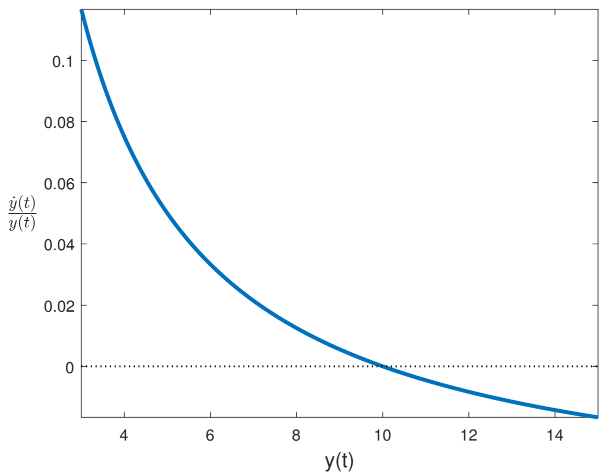
Beyond knowing that the economy converges to its steady state, we would also like to know what shape this convergence take. This allows us to answer questions like: What would we expect to happen to the growth rate of output per worker and the population after the introduction of the potato in Ireland. To understand the shape of convergence, consider again the, now slightly rewritten, solution for $y(t)$:

$$y(t) - \underbrace{\frac{1}{Z}}_{y^*} = \left[y(0) - \frac{1}{Z} \right] \exp(-\alpha Z t). \tag{25}$$

We directly see that $y(t) - y^*$ is an exponential growth process with growth rate $-\alpha Z$. Put differently, the distance between $y(t)$ and its steady state converges over to zero over time at rate $-\alpha Z$. For example, if $-\alpha Z = -0.03$, the absolute gap between $y(t)$ and its steady state vanishes each period by 3%. Figure 1 shows this behavior graphically for an economy that starts with an output per worker 50% above its steady state and another economy that starts with an output per worker 50% below its steady state. The vertical dotted lines going from $y^* = 10$ to each of the curves display $y(t) - y^*$ for each economy. As the figure highlights, this distance is indeed declining in an exponential fashion.

In economic growth, we usually find it useful to think in terms of growth rates

Figure 2: The growth rate of output per worker



instead of absolute changes in a variable. To study the growth rate of $y(t)$, we return to the differential equation for $y(t)$ and write it as a growth rate:

$$\dot{y}(t) = -\alpha Z y(t) + \alpha \tag{26}$$

$$\frac{\dot{y}(t)}{y(t)} = -\alpha Z + \frac{\alpha}{y(t)} \tag{27}$$

Note, the growth rate is 0 if the economy is in its steady state: $y(t) = \frac{1}{Z} = y^*$. Important for us, it is a decreasing, convex function in $y(t)$, and

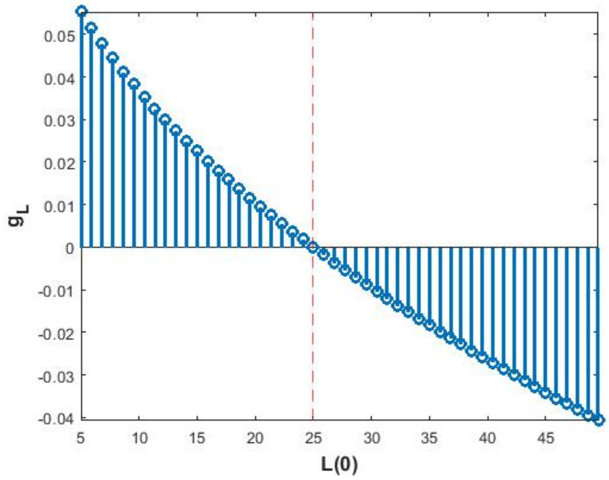
$$y(t) \mapsto 0 \quad \frac{\dot{y}(t)}{y(t)} \mapsto \infty$$

$$y(t) \mapsto \infty \quad \frac{\dot{y}(t)}{y(t)} \mapsto -\alpha Z.$$

The first equation says that as output per worker of an economy approaches zero, the growth rate of output per worker approaches infinity. The second equation tells us that as output per worker of an economy approaches infinity, the growth rate of output per worker approaches $-\alpha Z$

Figure 2 shows these properties graphically. It highlights that the (absolute) growth rate is higher the further the economy is away from steady state. That is, an economy that is below its steady state will see particular high output per

Figure 3: The growth rate of labor



worker growth rates the further it is below its steady state. Similarly, an economy will have a particularly high negative growth rate if it is far above its steady state.

Once we have the dynamics of output per worker, it is straightforward to derive those for labor. To that end, substitute $y(t) = AL(t)^{-\alpha}$ into the dynamics of output:

$$y(t) = \frac{1}{Z} - \left[\frac{1}{Z} - y(0)\right] \exp(-\alpha Z t) \tag{28}$$

$$AL(t)^{-\alpha} = \frac{1}{Z} - \left[\frac{1}{Z} - AL(0)^{-\alpha}\right] \exp(-\alpha Z t) \tag{29}$$

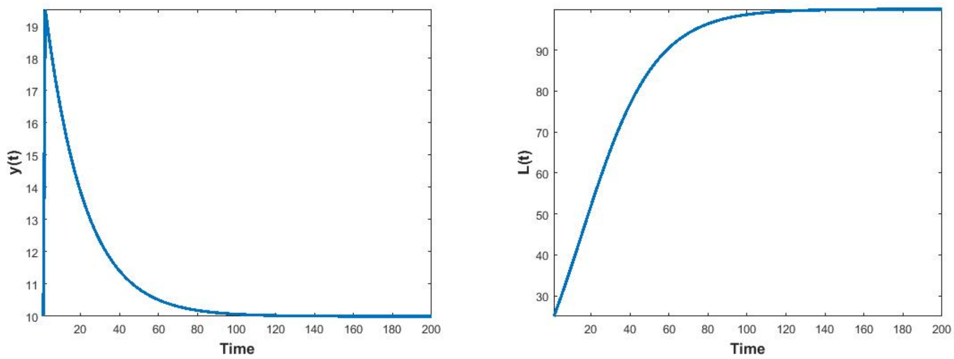
$$\frac{1}{L(t)^{\alpha}} = \frac{1}{AZ} + \left[\frac{1}{L(0)^{\alpha}} - \frac{1}{AZ}\right] \exp(-\alpha Z t) \tag{30}$$

Parallel to the conclusions for the dynamics of output per worker, we can derive for labor:

- $L(0) = AZ$, $L(t) = AZ \ \forall t$.
- Over time, $L(t)^{\alpha}$ converges to its steady state AZ for any $L(0)$.
- The gap between $\frac{1}{L(t)^{\alpha}}$ and its steady state vanishes at a constant rate αZ.

Figure 3 shows the resulting growth rate of labor over time for different levels of $L(t)$. It highlights that the population growth is the largest (lowest) the further

Figure 4: A one-time increase in productivity



below (above) the economy is from its steady state. The economic intuition directly follows from the link between labor and output per worker: An economy which labor is far below its steady state has output per worker far above its steady state and, hence, a high population growth rate. What may cause an economy to have a labor force well below its steady state? In the medieval UK, wars may have been one cause. Another cause may have been large one-time increases in productivity such as the introduction of the potato.

Figure 4 displays the dynamics of labor and output per worker after such a one-time increase in productivity. In period 0, labor is unchanged and, hence, output per worker is above its steady state as the left panel of the figure shows. As a result, population growth is positive, and labor starts growing as the right panel of the figure shows. The increase in labor depresses output per worker which starts falling back to its (unchanged) steady state level in the fashion we have seen before. While output per worker remains above its steady state level, labor keeps growing though the growth rate slows down as the economy converges to its new steady state level of labor.

1.3 Continuous productivity growth

We now turn to the case where productivity is constantly growing. This case is interesting for at least two reasons. First, it may be tempting to think that continuous productivity growth offers a way out of the poverty trap of the Malthus model. As the economy constantly becomes more productive, income per worker may grow over time. We will see that this intuition is, generally, wrong. Second, it is probably a better way to think about productivity changes. Though one could think of the introduction of the potato as one large change, it is probably true that the diffusion of this new technology was not instantaneous. Initially, only few farmers used the new technology, and many other farmers were still growing weed. Over time, the latter group learned from the former group and also adopted the technology leading to a slow diffusion over time. We will consider a constant exponential growth rate for technology:

$$A(t) = A(0)\exp(gt) \Rightarrow \frac{\dot{A}}{A} = g. \tag{31}$$

1.4 The steady state

We will procced as before: Assume a steady state exists where output per worker growth is zero. The growth rate of output per worker depends now on the growth rate of technology and the growth rate of labor:

$$y(t) = A(t)L(t)^{-\alpha} \tag{32}$$

$$\ln y(t) = \ln A(t) - \alpha \ln L(t) \tag{33}$$

$$\frac{\dot{y}(t)}{y(t)} = \frac{\dot{A}(t)}{A(t)} - \alpha\frac{\dot{L}(t)}{L(t)} \tag{34}$$

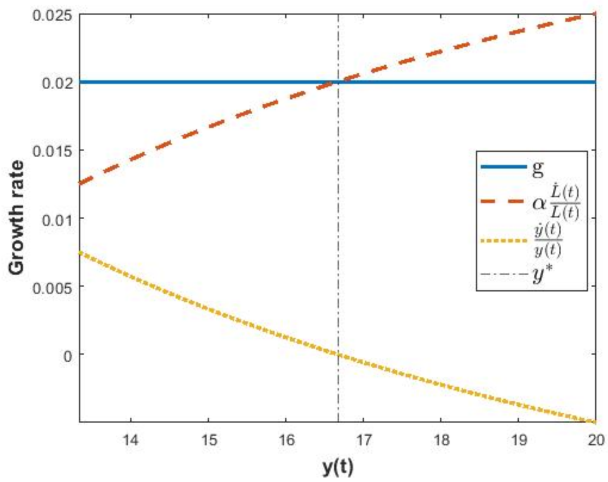
We have a steady state in output per worker when

$$g = \alpha\frac{\dot{L}(t)}{L(t)} = \alpha\left(Z - \frac{1}{y^*}\right) \tag{35}$$

$$y^* = \frac{\alpha}{\alpha Z - g}. \tag{36}$$

The steady state does, indeed, exist, i.e., the right-hand side is a constant. A naive look at the equation may suggest that the steady state output per worker is negative if $g > \alpha Z$. This naive conclusion is wrong. We have derived the equation

Figure 5: Steady state with productivity growth



assuming that a steady state exists with $\frac{\dot{y}(t)}{y(t)} = 0$. From

$$\frac{\dot{y}(t)}{y(t)} = \frac{\dot{A}(t)}{A(t)} - \alpha\frac{\dot{L}(t)}{L(t)} \tag{37}$$

it is obvious that no such steady state exists if $g > \alpha Z$. That is, if technological growth is fast enough such that even with the population growing at its natural rate output per worker is still growing, then permanent growth in output per worker will realize. However, as discussed above, a reasonable number for the natural birth rate may be 300% within a generation, and we have not recorded such high productivity growth in human history. Hence, this case seems empirically irrelevant, and we can conclude that the poverty trap of the Malthus Model (zero growth in output per worker in the long run) also holds with constant technological progress.

Having clarified that productivity growth will generally not lead to a positive output-per-worker *growth rate* in steady state, it is important to understand that productivity growth does affect the *level* of output per worker in steady state. Figure 5 shows this point graphically. In steady state, $g = \alpha\frac{\dot{L}(t)}{L(t)}$ which is the intersection of the blue line with the red dashed line. At that level of output per worker, the growth rate of output per worker (yellow dotted line) is zero. Note, different from the model with a constant level of productivity, labor is now growing in steady state at rate $\frac{g}{\alpha}$. At lower levels of output per worker (to the left of the steady state), output per worker will be growing, and labor will be growing at a rate below its steady state value. The reverse is true at levels of income per worker above the steady state value.

1.4.1 The dynamics of the model

We can proceed as with a constant level of technology to obtain a solution for $y(t)$. We have

$$\frac{\dot{y}(t)}{y(t)} = g - \alpha \frac{\dot{L}(t)}{L(t)} \tag{38}$$

$$\frac{\dot{y}(t)}{y(t)} = g - \alpha \left[Z - \frac{1}{y(t)} \right] \tag{39}$$

$$\dot{y}(t) = (-\alpha Z + g) y(t) + \alpha. \tag{40}$$

Define again an auxiliary variable $u(t)$ to get rid of the constant:

$$u(t) = \dot{y}(t) = -(\alpha Z - g) y(t) + \alpha \tag{41}$$

$$\dot{u}(t) = -(\alpha Z - g) \dot{y}(t) \tag{42}$$

$$\Rightarrow \dot{u}(t) = -(\alpha Z - g) u(t) \tag{43}$$

$$u(t) = u(0) \exp(-(\alpha Z - g) t) \tag{44}$$

Substituting again for $u(t)$ to obtain the solution for $y(t)$:

$$-(\alpha Z - g) y(t) + \alpha = [-(\alpha Z - g) y(0) + \alpha] \exp(-(\alpha Z - g) t) \tag{45}$$

$$y(t) = \underbrace{\frac{\alpha}{\alpha Z - g}}_{y^*} + \left[y(0) - \frac{\alpha}{\alpha Z - g} \right] \exp(-(\alpha Z - g) t). \tag{46}$$

The dynamics of output per worker are very similar to the case when technology is constant. Output per worker again converges to its steady state, and the distance between output per worker and its steady state follows again an exponential growth process that converges to zero. Technological progress changes two things about the dynamics of output per worker. First, as discussed, it increases the steady state level. Second, it affects the rate of convergence $\alpha Z - g$. A higher technological growth rate slows down the speed at which the economy converges to its steady state.

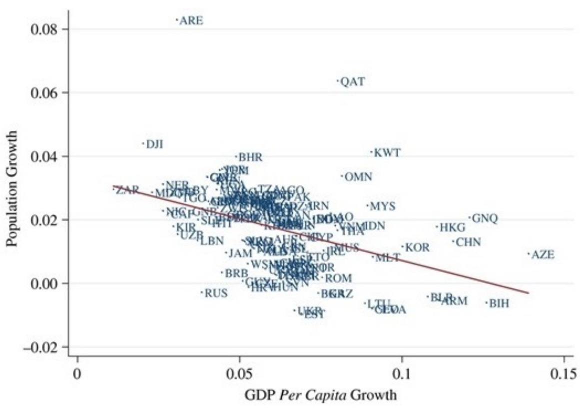
1.5 Policy implications and discussion

Malthus himself derived policy implications for his time. Using the logic of his ideas, he was against income support to the poor as he thought that any poverty reduction would only lead population growth without permanent improvements in economic well-being:

"*Yet in all societies, even those that are most vicious, the tendency to a virtuous attachment (i.e., marriage) is so strong, that there is a constant effort towards an increase of population. This constant effort as constantly tends to subject the lower classes of the society to distress and to prevent any great permanent amelioration of their condition.*"

Instead, Malthus saw the reduction in birth rates of the poor as the key to their economic development. Being a deeply religious person, he advocated for the postponement of marriage, and celibacy for poor people.

Figure 6: Population growth rate and income per person



Source: Brueckner and Schwandt (2015)

Before thinking about the policy implications for today's world, it is worth to ask about the relevancy of the model for today. Figure 6 shows that today, high income per person is rather associated with low population growth instead of high population growth. Today's East Asian countries are a good example. Many of these economies have seen rapid economic development over the last decades and, at the same time, saw their population growth rates plumbing. Put differently, the key mechanism of the Malthus model, income per worker driving population growth, seems to be no longer operative. The most likely explanation is the wide availability of birth control in most economies. Methods of contraception allow us today to choose any birth rate we desire. In all developed economies, this choice lies well below the natural birth rate.

Having said this, it may be premature to conclude that the Malthus model has no relevance for today's world any longer. The poorest economies do not look that different from the medieval UK. For example in Chad, 80% of the population work in agriculture, and 36.5% of the population live in absolute poverty. Moreover, less than 10% of married women in Chad report to use any method of contraception. Hence, in the spirit of Malthus, the United Nations included the use of contraceptives among their Sustainable Development Goals:
"*By 2030, ensure universal access to sexual and reproductive health-care services, including for family planning, information and education, and the integration of reproductive health into national strategies and programs.*."

References

Brueckner, M. and H. Schwandt (2015): "Income and population growth," *The Economic Journal*, 125, 1653–1676.

Malthus, T. R. (1798): *An essay on the principle of population*, J. Johnson.